Displacement field of a point force acting on the surface of an elastically anisotropic half-space

1994 J. Phys. A: Math. Gen. 27 7905

(http://iopscience.iop.org/0305-4470/27/23/033)

J. Phys. A: Math. Gen. 27 (1994) 7905–7914. Printed in the UK

Displacement field of a point force acting on the surface of an elastically anisotropic half-space

A G Every

Physics Department, University of the Witwatersrand, Johannesburg, PO WITS 2050, South Africa

Received 6 May 1994, in final form 7 October 1994

Abstract. A ring integral expression is presented for the Green tensor pertaining to the static displacement field of a point force acting on the surface of an elastically anisotropic half-space. It is derived as the low-frequency limit of the dynamic Green tensor. The expression is suitable for rapid computations, and illustrative numerical results are presented for a semi-infinite (001)-oriented silicon crystal. For surface displacements the Green tensor decomposes naturally into symmetric and antisymetric parts. The ring integral for the symmetric part can be performed analytically, yielding an algebraic result. Simplifications brought about by material symmetry are discussed.

1. Introduction

Green function methods are widely used in treating static and dynamic problems in elasticity. Applications include calculating the strain fields surrounding dislocations, cracks, inclusions, voids and other imperfections in bounded and infinitely extended elastic solids, and using this information to determine the interactions between individual defects and their interaction with boundaries [1]. Green functions describing the displacement response of solids to various forms of external loading are of direct relevance to indentation problems in mechanical testing and civil engineering calculations. There is also considerable interest at present in elastodynamic Green functions [2–6].

Two-dimensional plane strain problems are elegantly handled by the Stroh formalism, and have received considerable attention (see, for example, the papers by Ting and other authors in [7]). There has also been progress on a broad front in the solution of three-dimensional problems. As early as the last century explicit closed-form expressions were derived by Kelvin [8] for the displacement field of a point force in an infinite isotropic elastic continuum, and by Boussinesq [9] for the displacement field of an isotropic half-space acted on by a force at its boundary. In this century these results have been generalized in various ways by a number of authors. Mindlin [10] solved the problem of the displacement response of an isotropic half-space to a buried force, and Rongved [11] treated the corresponding problem for two joined elastic half-spaces. Layered media have been treated by a number of authors (see [12] and references contained therein), and there have been generalizations of these results by Pan and Chou [13] and others to a transversely isotropic half-space with the zonal axis normal to the surface.

0305-4470/94/237905+10$19.50

The Green functions of anisotropic elasticity are seldom found expressed in a closed-form in terms of elementary functions. They are usually obtained by integral transform methods, and there are difficulties in evaluating the inverse transform completely by analytical means because one arrives at an integrand involving the roots of a sextic equation. We exclude from this generalization transverse isotropy, which in many respects is more akin to isotropy than to general anisotropy, and other cases of special combinations of elastic constants for which the sextic equation can be factored. For certain applications, in particular where the spatial Fourier transform of the force distribution is conveniently at hand, a Fourier integral representation of the Green tensor is a common starting point for calculations. A three-dimensional integral representation of the Green tensor for the infinite anisotropic elastic continuum has been widely used by Mura and others [1]. Recently Walker [14] has derived an expression for the Green tensor of an anisotropic half-space acted on by a buried force. His result takes the form of a combination of a triple Fourier integral, which is the Green function for the infinite continuum, together with a quadruple Fourier integral, which describes the displacement field, which, when combined with the first, leaves the surface traction-free.

For numerical evaluation of the Green tensor, an integral representation of lower dimension is desirable. Lifshitz and Rosenzweig [15] first showed that the Green tensor for the infinite anisotropic continuum could be expressed in the form of a one-dimensional ring integral taken around the observation direction, and variations of this result have been reported by a number of other authors since then [6, 16, 17]. The frequency and time domain dynamic Green tensors are expressible as a combination of a surface integral over the unit hemisphere centred on the source–receiver direction together with a ring integral around the periphery of this hemisphere, which reduces to the ring integral in the static limit [5, 6]. Willis [18] has formally solved a broad class of self-similar problems in elastodynamics, including the dynamic response of an anisotropic half-space to surface loading which can be expressed as a homogeneous function of position and time.

In this paper we derive an expression for the static displacement field of an anisotropic half-space subjected to a static concentrated point force acting on the surface. The expression takes the form of a one-dimensional ring integral, and is suitable for rapid numerical computations. It is obtained as the low-frequency limit to the corresponding frequency domain dynamical Green tensor. Our method has certain features in common with that of Willis [18], although we are not treating the homogeneous problem. Our derivation could be simplified by framing it in terms of purely elastostatic considerations, but we have obtained it in the course of investigations into the dynamic behaviour of anisotropic solids, and it seems to us that there is some value in presenting the derivation from this perspective. We express the dynamic Green tensor as a surface integral over all values of slowness $s_{\|}$ parallel to the surface of the half-space. In the static or low-frequency limit the integration with respect to the magnitude of $s_{\|}$ can be done analytically. The remaining integral over the direction of $s_{\|}$, in general, requires numerical methods for its evaluation, since the integrand is an algebraic expression involving the roots of a sextic equation. For surface displacements there is some simplification in that the Green tensor decomposes naturally into symmetric and antisymmetric parts. The ring integral for the symmetric part can be performed analytically, yielding an algebraic result. By way of a numerical example, we compare Green functions for a semi-infinite (100)-oriented silicon crystal with those of the infinite continuum.

2. Formulation of the problem

We consider an anisotropic elastic half-space occupying the domain $x_3 > 0$. A time-harmonic concentrated point force

$$F_j(\boldsymbol{x}_\parallel, t) = f_j \delta(\boldsymbol{x}_\parallel) \mathrm{e}^{-\mathrm{i}\omega t} \tag{1}$$

of frequency ω acts on the surface of the half-space at the origin. f_j are the Cartesian components of the amplitude of this force and $\delta(\boldsymbol{x}_\parallel) = \delta(x_1)\delta(x_2)$ is the two-dimensional delta function (see figure 1). Within the half-space the displacement field $\boldsymbol{u}(\boldsymbol{x})\mathrm{e}^{-\mathrm{i}\omega t}$ is governed by the Christoffel equations [19]

$$\left(C_{r\ell sm} \frac{\partial^2}{\partial x_\ell \partial x_m} + \rho\omega^2 \delta_{rs} \right) u_s = 0 \tag{2}$$

where $C_{r\ell sm}$ is the elastic modulus tensor for the medium and ρ is its density. A solution of (2) is required that satisfies the boundary conditions, i.e. yields a traction force at the surface equal and opposite to the applied load, and only consists of out-going waves, i.e. either homogeneous waves with ray vectors directed into the half-space or inhomogeneous waves which decay exponentially into the half-space (evanescent waves). It is in fact only the latter that will concern us as ω is allowed to go to zero.

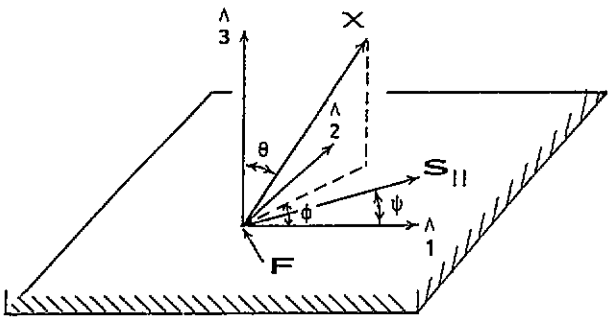

Figure 1. Coordinate system used in calculations.

We proceed by expressing the delta function in (1) in terms of its Fourier transform, so that

$$F_j(\boldsymbol{x}_\parallel, t) = \frac{f_j}{(2\pi)^2} \int \mathrm{d}k_1\, \mathrm{d}k_2\, \mathrm{e}^{\mathrm{i}(\boldsymbol{k}_\parallel \cdot \boldsymbol{x}_\parallel - \omega t)} \tag{3}$$

where $\boldsymbol{k}_\parallel = (k_1, k_2)$ is the component of the wavevector parallel to the surface. The response to each Fourier component of the force is a linear superposition of three out-going plane waves which phase match in the surface with that component, giving a displacement field of the form

$$u_i(\boldsymbol{k}_\parallel; \boldsymbol{x}, t) = \sum_{n=1}^{3} \Gamma_n U_i^{(n)} \mathrm{e}^{\mathrm{i}(\boldsymbol{k}^{(n)} \cdot \boldsymbol{x} - \omega t)} \tag{4}$$

where $\boldsymbol{k}^{(n)} = (k_1, k_2, k_3^{(n)})$, $\boldsymbol{U}^{(n)}$ and Γ_n are the wavevectors, polarization vectors and amplitudes of the three out-going waves labelled by the index n, respectively. The $\boldsymbol{U}^{(n)}$ are the solutions of [19]

$$\left(C_{r\ell sm} s_\ell^{(n)} s_m^{(n)} - \rho \delta_{rs} \right) U_s^{(n)} = 0 \tag{5}$$

where $s^{(n)} = k^{(n)}/\omega = \left(s_1, s_2, s_3^{(n)}\right)$ are the acoustic slownesses of the three waves. $s_3^{(n)}$, for given $s_\parallel = k_\parallel/\omega$, are the three roots of the characteristic equation

$$|C_{r\ell sm} s_\ell s_m - \rho\delta_{rs}| = 0 \tag{6}$$

which correspond to out-going waves. The remaining three roots of this sextic equation are discarded as they correspond to incoming or exponentially diverging waves, which do not satisfy the boundary conditions. For $|s_\parallel| \gg |\rho/C_{r\ell sm}|$ the three $s_3^{(n)}$ are complex.

The strain field that $u_i(k_\parallel; x, t)$ gives rise to is

$$\epsilon_{pq} = \frac{1}{2}\left(\frac{\partial u_p}{\partial x_q} + \frac{\partial u_q}{\partial x_p}\right) = \frac{\mathrm{i}}{2}\sum_{n=1}^{3}\Gamma_n\left\{U_p^{(n)}k_q^{(n)} + U_q^{(n)}k_p^{(n)}\right\}\mathrm{e}^{\mathrm{i}(k^{(n)}\cdot x - \omega t)} \tag{7}$$

and the traction force at the surface is given by the associated stress components

$$\sigma_{j3}(x_3 = 0) = \sum_{p,q=1}^{3} C_{j3pq}\epsilon_{pq}(x_3 = 0) = \mathrm{i}\omega\sum_{n=1}^{3}\Gamma_n A_{jn}\mathrm{e}^{\mathrm{i}(k_\parallel\cdot x_\parallel - \omega t)} \tag{8}$$

where

$$A_{jn} = \sum_{p,q=1}^{3} C_{j3pq}U_p^{(n)}s_q^{(n)}\,. \tag{9}$$

The symmetry of C_{i3pq} with respect to the interchange of p and q has been used to equate the two terms in (7) thereby eliminating the factor of $\frac{1}{2}$. The traction force is equal and opposite to the corresponding Fourier component of the external force acting on the surface (equation (3)), from which it follows that

$$\Gamma_n = \frac{-1}{4\pi^2\mathrm{i}\omega}\sum_{j=1}^{3}\left(A^{-1}\right)_{nj}f_j\,. \tag{10}$$

Combining (4) and (10) and integrating over $k_\parallel$, the displacement field due to $F(x_\parallel, t)$ is found to be

$$u_i(x, t) = \sum_{j=1}^{3} G_{ij}(x, \omega)f_j\mathrm{e}^{-\mathrm{i}\omega t} \tag{11}$$

where

$$G_{ij}(x, \omega) = \frac{-1}{4\pi^2\mathrm{i}\omega}\sum_{n=1}^{3}\iint \mathrm{d}k_1\,\mathrm{d}k_2\left(A^{-1}\right)_{nj}U_i^{(n)}\mathrm{e}^{\mathrm{i}k^{(n)}\cdot x} \tag{12}$$

is the frequency domain dynamic Green tensor. On changing the integration variable to $s_\parallel = k_\parallel/\omega$ one obtains

$$G_{ij}(x, \omega) = \frac{\mathrm{i}\omega}{4\pi^2}\sum_{n=1}^{3}\iint \mathrm{d}s_1\,\mathrm{d}s_2\,Q_{ij}^{(n)}(s_\parallel)\mathrm{e}^{\mathrm{i}\omega s^{(n)}\cdot x} \tag{13}$$

where

$$Q_{ij}^{(n)}(s_\parallel) = \left(A^{-1}\right)_{nj}U_i^{(n)}\,. \tag{14}$$

The behaviour of $Q_{ij}^{(n)}(s_\parallel)$ for small $s_\parallel$ is fairly complicated, displaying *inter alia* a pole at the Rayleigh slowness, but this will not concern us here, since we will make use only of the relatively simple limiting behaviour of $Q_{ij}^{(n)}(s_\parallel)$ as $s_\parallel \to \infty$.

3. Static limit

We obtain the static Green tensor $G_{ij}(\boldsymbol{x})$ by taking the limit $\omega \to 0$ of $G_{ij}(\boldsymbol{x}, \omega)$:

$$G_{ij}(\boldsymbol{x}) = \lim_{\omega \to 0} \frac{\mathrm{i}\omega}{4\pi^2} \sum_{n=1}^{3} \iint \mathrm{d}s_1\, \mathrm{d}s_2\, Q_{ij}^{(n)}(\boldsymbol{s}_\parallel) \mathrm{e}^{\mathrm{i}\omega \boldsymbol{s}^{(n)} \cdot \boldsymbol{x}} . \tag{15}$$

We proceed by dividing the domain of integration into two portions, one contained within a circle centred on the origin and of radius s° which is large compared to the Rayleigh slowness s_{R} in any direction, and the other lying outside this circle. The integral over the inner domain is finite, and because of the factor $\mathrm{i}\omega$ in (15), its contribution to $G_{ij}(\boldsymbol{x})$ vanishes in the low-frequency limit. We therefore need only consider the integral over the outer domain. A change of variable to s and ψ, where $s_1 = s\cos(\psi)$ and $s_2 = s\sin(\psi)$, yields

$$G_{ij}(\boldsymbol{x}) = \lim_{\omega \to 0} \frac{\mathrm{i}\omega}{4\pi^2} \sum_{n=1}^{3} \int_0^{2\pi} \mathrm{d}\psi \int_{s^{\circ}}^{\infty} \mathrm{d}s\, s Q_{ij}^{(n)}(s, \psi) \mathrm{e}^{\mathrm{i}\omega(sx_1\cos(\psi) + sx_2\sin(\psi) + s_3^{(n)}(s,\psi)x_3)} . \tag{16}$$

The limiting behaviour of $s_3^{(n)}(s, \psi)$ and $Q_{ij}^{(n)}(s, \psi)$ for large s allows the integral over s to be done analytically. For $s > s^{\circ}$ all terms in the slowness equation except those of degree 6 in s and s_3 can be neglected. This is equivalent to taking the density to zero, and transforms the elastodynamic problem into an elastostatics one. By dividing through by s, one arrives at a sextic equation for s_3/s, whose coefficients are functions of the elastic constants and ψ only. The roots all occur in complex conjugate pairs, and for out-going waves the three roots are chosen whose imaginary parts are positive and thus correspond to inhomogeneous waves which die off exponentially into the half-space. These roots can be expressed in the form

$$s_3^{(n)} = \left(\alpha^{(n)}(\psi) + \mathrm{i}\beta^{(n)}(\psi)\right)s \tag{17}$$

where $\alpha(\psi)$ and $\beta(\psi)$ have the cyclic properties

$$\alpha^{(n)}(\psi + \pi) = -\alpha^{(n)}(\psi) \qquad \beta^{(n)}(\psi + \pi) = \beta^{(n)}(\psi) \tag{18}$$

so that $\boldsymbol{s}^{(n)} \to -\boldsymbol{s}^{(n)*}$ and an out-going wave is retained on the reversal of $\boldsymbol{s}_\parallel$.

Equation (5) shows on reversal of $\boldsymbol{s}_\parallel$ that $\boldsymbol{U}^{(n)} \to \boldsymbol{U}^{(n)*}$, equation (9) shows that $A_{jn} \to -A_{jn}^*$ and hence $\left(A^{-1}\right)_{nj} \to -\left(A^{-1}\right)_{nj}^*$, and (14) shows that $Q_{ij}^{(n)} \to -Q_{ij}^{(n)*}$. Furthermore, $Q_{ij}^{(n)}$ is unaffected by the normalization of the $\boldsymbol{U}^{(n)}$, and for large s is inversely proportional to s, and so $sQ_{ij}^{(n)}$ is independent of s and can be expressed in the form

$$sQ_{ij}^{(n)} = a_{ij}^{(n)}(\psi) + \mathrm{i}b_{ij}^{(n)}(\psi) \tag{19}$$

where $a_{ij}^{(n)}(\psi)$ and $b_{ij}^{(n)}(\psi)$ have the cyclic properties

$$a_{ij}^{(n)}(\psi + \pi) = -a_{ij}^{(n)}(\psi) \qquad b_{ij}^{(n)}(\psi + \pi) = b_{ij}^{(n)}(\psi) . \tag{20}$$

On substituting these results into (16) and carrying out a change of variable from s to $y = s\omega\beta^{(n)}x_3$, we obtain

$$G_{ij}(x) = \lim_{\omega \to 0} \frac{1}{4\pi^2 x_3} \sum_{n=1}^{3} \int_0^{2\pi} \mathrm{d}\psi \frac{\left(\mathrm{i}a_{ij}^{(n)} - b_{ij}^{(n)}\right)}{\beta^{(n)}} \int_{s^{\circ}\omega\beta^{(n)}x_3}^{\infty} \mathrm{d}y\, \mathrm{e}^{y(\mathrm{i}\gamma^{(n)} - 1)} \tag{21}$$

where

$$\gamma^{(n)} = \frac{x_1\cos(\psi) + x_2\sin(\psi) + \alpha^{(n)}x_3}{\beta^{(n)}x_3} = \frac{\sin(\theta)\cos(\psi-\phi) + \alpha^{(n)}\cos(\theta)}{\beta^{(n)}\cos(\theta)} \tag{22}$$

where θ and ϕ are the polar angles describing the direction of $\boldsymbol{x}$. The $\gamma^{(n)}$ satisfy the condition

$$\gamma^{(n)}(\psi+\pi) = -\gamma^{(n)}(\psi). \tag{23}$$

As $\omega \to 0$, the lower limit of the integral with respect to y goes to zero, and this integral can be done analytically, leading to the result

$$G_{ij}(\boldsymbol{x}) = \frac{1}{4\pi^2 x_3}\sum_{n=1}^{3}\int_0^{2\pi} \mathrm{d}\psi \frac{\left(\mathrm{i}a_{ij}^{(n)} - b_{ij}^{(n)}\right)}{\beta^{(n)}} \frac{\left\{1+\mathrm{i}\gamma^{(n)}\right\}}{\left\{1+(\gamma^{(n)})^2\right\}}. \tag{24}$$

Because of the cyclic properties of $a_{ij}^{(n)}$, $b_{ij}^{(n)}$, $\beta^{(n)}$ and $\gamma^{(n)}$, the imaginary part of the integrand changes sign on advancing θ by π, and so integrates out to zero. The real part of the integrand returns to the same value on advancing ψ by π, and so the integral may be restricted to the interval $[0, \pi]$ and doubled. The Green tensor is thus real, and is given by

$$G_{ij}(\boldsymbol{x}) = \frac{-1}{2\pi^2 x_3}\sum_{n=1}^{3}\int_0^{\pi} \mathrm{d}\psi \frac{\left\{a_{ij}^{(n)}\gamma^{(n)} + b_{ij}^{(n)}\right\}}{\beta^{(n)}\left\{1+(\gamma^{(n)})^2\right\}}. \tag{25}$$

4. Surface displacements

In order to calculate the surface displacement, the limiting behaviour of the integral in (25) as $x_3 \to 0$ has to be ascertained and this, as we see below, leads to the cancellation of the divergence associated with the factor $1/x_3$ in front of the sum. Near the surface $\theta \approx \pi/2$, so $\sin\theta \approx 1$ and $0 < \cos\theta \ll 1$, and so from (22)

$$\gamma^{(n)} \approx \frac{\cos(\psi-\phi)}{\beta^{(n)}\cos\theta} \tag{26}$$

and is very large except where $\cos(\psi - \phi) \approx 0$.

In the surface, $G_{ij}(\boldsymbol{x}_\parallel)$ may be decomposed into symmetric and antisymmetric parts, given, respectively, by

$$G_{ij}^{\mathrm{S}}(\boldsymbol{x}_\parallel) = \tfrac{1}{2}(G_{ij}(\boldsymbol{x}_\parallel) + G_{ji}(\boldsymbol{x}_\parallel)) = \lim_{\cos\theta\to 0} \frac{-1}{2\pi^2|\boldsymbol{x}_\parallel|\cos\theta}\sum_{n=1}^{3}\int_0^{\pi}\mathrm{d}\psi \frac{b_{ij}^{(n)}}{\beta^{(n)}\left\{1+(\gamma^{(n)})^2\right\}} \tag{27}$$

$$G_{ij}^{\mathrm{A}}(\boldsymbol{x}_\parallel) = \tfrac{1}{2}(G_{ij}(\boldsymbol{x}_\parallel) - G_{ji}(\boldsymbol{x}_\parallel)) = \lim_{\cos\theta\to 0} \frac{-1}{2\pi^2|\boldsymbol{x}_\parallel|\cos\theta}\sum_{n=1}^{3}\int_0^{\pi}\mathrm{d}\psi \frac{a_{ij}^{(n)}\gamma^{(n)}}{\beta^{(n)}\left\{1+(\gamma^{(n)})^2\right\}}. \tag{28}$$

This follows from Betti's reciprocal work theorem, which for points in the surface implies that $G_{ij}(\boldsymbol{x}_\parallel) = G_{ji}(-\boldsymbol{x}_\parallel)$. Reversing the direction of $\boldsymbol{x}_\parallel$ is equivalent to advancing ϕ by π, and according to (26) this changes the sign of $\gamma^{(n)}$, while $a_{ij}^{(n)}$, $b_{ij}^{(n)}$ and $\beta^{(n)}$ are unaffected, since they do not depend on $\boldsymbol{x}$. In the limit as $\cos\theta \to 0$, the only contribution to $G_{ij}^{\mathrm{S}}(\boldsymbol{x}_\parallel)$ arises from integration over a small interval around $\psi = \phi + \pi/2$ or $\phi - \pi/2$, whichever lies in $[0, \pi]$. Here we may take $a_{ij}^{(n)}$, $b_{ij}^{(n)}$ and $\beta^{(n)}$ to be constant, and approximate $\cos(\psi-\phi) \approx \psi-\phi\pm\pi/2$, so that $\gamma^{(n)}$ is linear in ψ. On changing the integration variable to $\gamma^{(n)}$, $\beta^{(n)}\cos\theta$ cancels out and we obtain

$$G_{ij}^{\mathrm{S}}(\boldsymbol{x}_{\|}) = \frac{-1}{2\pi^2|\boldsymbol{x}_{\|}|}\sum_{n=1}^{3} b_{ij}^{(n)} \int \frac{\mathrm{d}\gamma^{(n)}}{(1+(\gamma^{(n)})^2)} . \tag{29}$$

The integral is equal to π and so

$$G_{ij}^{\mathrm{S}}(\boldsymbol{x}_{\|}) = \frac{-1}{2\pi|\boldsymbol{x}_{\|}|}\sum_{n=1}^{3} b_{ij}^{(n)}(\psi=\phi\pm\pi/2) . \tag{30}$$

Thus, $G_{ij}^{\mathrm{S}}(\boldsymbol{x}_{\|})$ and hence the diagonal components of $G_{ij}(\boldsymbol{x}_{\|})$ are determined algebraically from the values of $b_{ij}^{(n)}$ in the direction $\psi = \phi \pm \pi/2$, and as expected are inversely proportional to $|\boldsymbol{x}| = |\boldsymbol{x}_{\|}|$.

The antisymmetric component is given by

$$\begin{aligned} G_{ij}^{\mathrm{A}}(\boldsymbol{x}_{\|}) &= \lim_{\cos\theta\to 0}\frac{-1}{2\pi^2|\boldsymbol{x}_{\|}|}\sum_{n=1}^{3}\int_0^{\pi}\mathrm{d}\psi \frac{a_{ij}^{(n)}\cos(\psi-\phi)}{\left\{(\beta^{(n)}\cos\theta)^2+(\cos(\psi-\phi))^2\right\}} \\ &= \frac{-1}{2\pi^2|\boldsymbol{x}_{\|}|}\sum_{n=1}^{3}\mathcal{P}\int_0^{\pi}\mathrm{d}\psi\frac{a_{ij}^{(n)}}{\cos(\psi-\phi)} \end{aligned} \tag{31}$$

where $\mathcal{P}$ denotes that the principal part is taken on integration through $\psi = \phi \pm \pi/2$.

In general, G_{ij}^{S} has six independent components and G_{ij}^{A} has three, but the existence of material symmetry can reduce these numbers. Thus, for example, if the (100) and (010) planes are symmetry planes (orthorhombic, tetragonal, hexagonal and cubic symmetry and isotropy permit this) then the following relations hold on reflecting through the (100) and (010) planes and applying Betti's theorem:

$$\begin{aligned} G_{13}(x_1,x_2) &= -G_{13}(-x_1,x_2) = -G_{13}(-x_1,-x_2) = -G_{31}(x_1,x_2) \\ G_{23}(x_1,x_2) &= -G_{23}(x_1,-x_2) = -G_{23}(-x_1,-x_2) = -G_{32}(x_1,x_2) \\ G_{12}(x_1,x_2) &= -G_{12}(-x_1,x_2) = +G_{12}(-x_1,-x_2) = +G_{21}(x_1,x_2) . \end{aligned} \tag{32}$$

It follows from these relations that $G_{13}^{\mathrm{S}} = G_{23}^{\mathrm{S}} = 0$ and $G_{12}^{\mathrm{A}} = 0$. There is further reduction for $\boldsymbol{x}_{\|}$ in a symmetry plane. Thus for $x_2 = 0$, $G_{23}^{\mathrm{A}} = 0$ and $G_{12}^{\mathrm{S}} = 0$.

Wu *et al* [20] have derived an analogous result to (30) and (31) for two-dimensional elasticity, reducing the symmetrical component of the surface displacement gradient due to a distribution of line forces to an algebraic expression, and the antisymmetrical part to a one-dimensional integral.

5. Numerical example

We have used (25) as the basis for numerical calculations on anisotropic as well as isotropic solids, using the trapezoidal rule for the integration. For isotropic solids the numerical results are in agreement with well known algebraic formulae found in Landau and Lifshitz [21] and elsewhere.

Figure 2 shows the variation of the Green tensor components G_{11} and G_{33} with direction in the (010)-plane of a semi-infinite silicon crystal with surface parallel to the (001) crystallographic plane, compared with the corresponding components G_{11}^{∞} and G_{33}^{∞} of

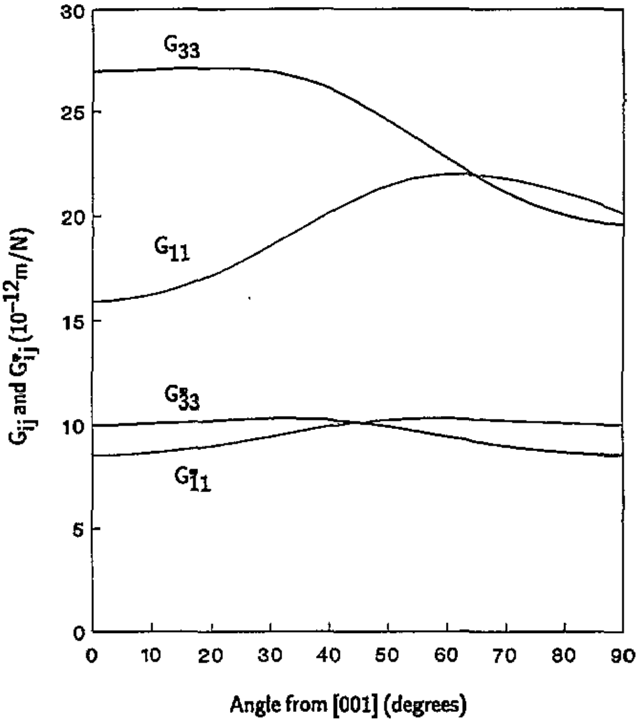

Figure 2. Angular dependence of G_{11} and G_{33} in the (010)-plane of a silicon half-space with (001)-oriented boundary, compared with the corresponding components G_{11}^{∞} and G_{33}^{∞} for the infinite continuum. Elastic constants for the calculation are from [22].

the Green tensor of the infinite continuum. The latter has been calculated using the ring integral expression from [6]

$$G_{ij}^{\infty}(x) = \frac{1}{8\pi^2\rho|x|}\sum_{n=1}^{3}\int_0^{2\pi} \mathrm{d}\psi\, (s^{(n)})^2 \Lambda_{ij}^{(n)} \tag{33}$$

where the integral is taken with respect to the direction in the plane perpendicular to x, and $\Lambda_{ij}^{(n)} = U_i^{(n)} U_j^{(n)}$. The half-space Green functions, as expected, are about a factor of two larger than the infinite continuum ones. Another important difference is that while $G_{11}^{\infty}(\theta) = G_{33}^{\infty}(90° - \theta)$ because of the material symmetry, the corresponding components of G_{ij} are not related in this way, because for the half-space the x_1- and x_3-axes are not equivalent. We also note that, because the (001)-plane is a symmetry plane of the infinite continuum, G_{11}^{∞} and G_{33}^{∞} are unchanged by reflection through that plane, implying that $\mathrm{d}G_{11}^{\infty}/\mathrm{d}\theta \to 0$ and $\mathrm{d}G_{33}^{\infty}/\mathrm{d}\theta \to 0$ as $\theta \to 90°$. This is not true for the half-space, as evident in figure 2, although the limiting value of $\mathrm{d}G_{33}/\mathrm{d}\theta$ is fairly small in this particular example.

Figure 3 shows the angular dependence of G_{ij} in the (001)-oriented surface of a silicon half-space. G_{ij} retains the 4-fold symmetry about the [001]-axis and the (100), (110), $(1\bar{1}0)$, and (010) symmetry planes. The symmetry operations have to be applied both to the angle and to the subscripts of G_{ij}, and thus, for example, $G_{33}(\theta) = G_{33}(90° - \theta)$, $G_{11}(\theta) = G_{22}(90° - \theta)$ and $G_{13}(\theta) = G_{23}(90° - \theta)$.

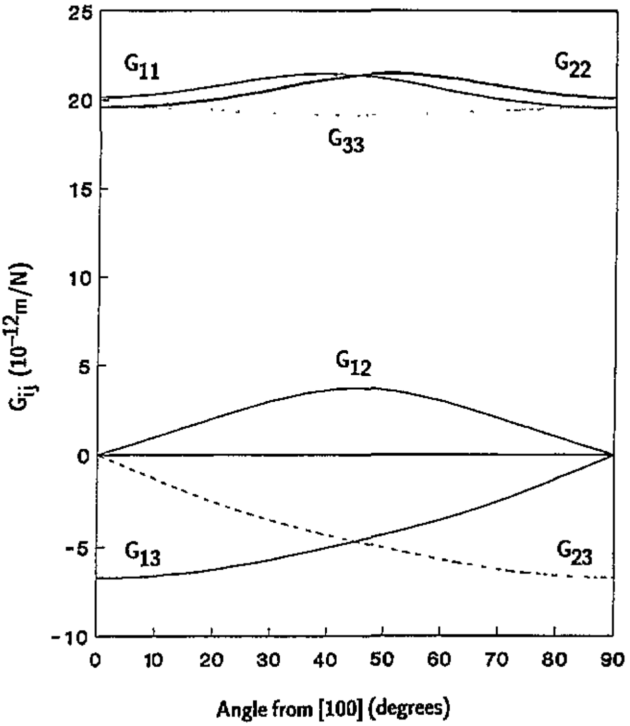

Figure 3. Angular dependence of G_{ij} in the (001)-oriented surface of a silicon half-space.

Acknowledgments

The author would like to express his appreciation for the hospitality of the Institute of Applied Mechanics, National Taiwan University, where this work was carried out. The NSC of the ROC and the South African FRD are thanked for financial support. Y H Pao, T T Wu, K C Wu, C H Chen and R Weaver are thanked for stimulating discussions. F R N Nabarro is thanked for drawing the author's attention to reference [14].

References

[1] Mura T 1982 *Micromechanics of Defects in Solids* (The Hague: Nijhoff)
[2] Burridge R, Chadwick P and Norris A N 1993 *Proc. R. Soc.* A **440** 655
[3] Tewary V K and Fortunko C M 1992 *J. Acoust. Soc. Am.* **91** 1888
[4] Kim K Y, Sachse W and Every A G 1993 *Phys. Rev. Lett.* **70** 3443
[5] Kim K Y, Every A G and Sachse W 1994 *J. Acoust. Soc. Am.* at press
[6] Every A G and Kim K Y 1994 *J. Acoust. Soc. Am.* at press
[7] Wu J J, Ting T C T and Barnett D M (eds) 1991 *Modern Theory of Anisotropic Elasticity and Applications* (Philadelphia, PA: SIAM)
[8] Thompson W 1848 *Camb. Dublin Math. J.* (February) p76
[9] Boussinesq J 1885 *Application des Potentiels a l'Etude de l'Equilibre et du Mouvement des Solides Elastiques* (Paris: Gauthier-Villars)
[10] Mindlin R D 1953 *Proc. 1st Midwestern Conf. on Solid Mechanics* p 56
[11] Rongved L 1955 *Proc. 2nd Midwestern Conf. on Solid Mechanics* p 1
[12] Benitez F G and Rosakis A J 1987 *J. Elasticity* **18** 3

[13] Pan Y C and Chou T W 1979 *Int. J. Eng. Sci.* **17** 545
[14] Walker K P 1993 *Proc. R. Soc.* A **443** 367
[15] Lifshitz I M and Rosenzveig L N 1947 *Z. Eksp. Theo. Fiz.* **17** 783
[16] Synge J L 1957 *The Hypercircle in Mathematical Physics* (Cambridge: Cambridge University Press)
[17] Barnett D M 1972 *Phys. Status Solidi* b **49** 741
[18] Willis J R 1973 *Phil. Trans. R. Soc.* **274** 435
[19] Musgrave M J P 1970 *Crystal Acoustics* (San Francisco, CA: Holden Day)
[20] Wu K C, Chiu Y T and Hwu Z H 1992 *ASME J. Appl. Mech.* **59** 344
[21] Landau L D and Lifshitz E M 1986 *Theory of Elasticity* (Oxford: Pergamon)
[22] Every A G and McCurdy A K 1992 *Numerical Data and functional Relationships in Science and Technology, New Series* vol 29, Group III *Landolt-Börnstein* ed D F Nelson (Berlin: Springer)